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Sandwich Theorems for Operators of Convex Type

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This note centers around a class of operators from a real vector space into a Dedekind complete ordered real vector space, which satisfy the usual convexity inequalities only with respect to some given subset Λ of the unit interval. It is shown that between a Λ-concave and a Λ-convex operator one can always find a Λ-affine operator. Several applications of this sandwich principle are given.

1. INTRODUCTION

In a number of recent applications of the Hahn–Banach and sandwich theorems, it turned out to be essential to relax the standard convexity assumptions from classical convex analysis. This aspect initiated the development of various new versions of these theorems which, at the same time, have simple shape and admit fast and widespread applications. The monograph of Fuchssteiner and Lusky [6] provides an excellent account of one sort of such results, namely sandwich type theorems for semigroups and cones. Another class of particularly useful results in this direction are the Hahn–Banach type theorems due to Fuchssteiner and König [5], which date back to early work of Fan [4] and König [10]. These results may be viewed as natural extensions of the classical interpolation theorem due to Mazur and Orlicz [14] and have become a flexible and powerful tool in different branches of analysis and its applications; we refer to Chapter II of [11] for a detailed discussion and a list of further references. Unfortunately, the proof of the main result of [5] is not as convincingly simple as one might expect from a result of such basic importance. Moreover, the methods from [5] do not extend to the case of vector–valued operators rather than real–valued functionals.

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In this note, we present a novel approach to more general results of this type, which avoids the cumbersome reduction to the classical case of sublinear operators. Instead the emphasis is here on the sandwich theory for a larger class of operators, which naturally generalized the midpoint convex operators with values in a Dedekind complete ordered vector space. One of our main results is Theorem 1, which provides the sandwich principle for such operators and has a remarkably short and simple proof. Among the consequences of this principle, we record, in Theorem 7, a vector–valued generalization of the Fuchssteiner–König theorem [5], which also contains a recent result of Martellotti and Salvadori [13] as a special case. In the final section, we collect some applications to the Choquet theory for families of upper semicontinuous functions on a compact Hausdorff space. In particular, we obtain, in Theorem 9 and Corollary 10, the existence and a characterization of the Shilov boundary under fairly mild convexity assumptions.

2. A Sandwich Theorem for Λ-Convex Operators

Throughout this section, let E be a real vector space, and let F be a Dedekind complete ordered real vector space. Thus F is ordered by a proper positive cone so that every non-empty subset of F which is bounded from below has an infimum in F, but we do not require F to be a lattice. As usual, it will be convenient to adjoin a smallest element $-\infty$ to the space F and to extend the algebraic operations from F to $F_* := F \cup \{-\infty\}$ by defining $0 \cdot (-\infty) := 0$, $t \cdot (-\infty) := -\infty$, and $x + (-\infty) := -\infty$ for all real $t > 0$ and all $x \in F$.

Given a subset Λ of the unit interval $[0, 1]$, a subset K of E is called Λ-*convex* if $\lambda\, u + (1 - \lambda)\, v \in K$ for all $u, v \in K$ and all $\lambda \in \Lambda$. If K is a Λ-convex subset of E, then an operator $T\colon K \rightarrow F_*$ is said to be Λ-*convex* if $T(\lambda\, u + (1 - \lambda)\, v) \leq \lambda\, T(u) + (1 - \lambda)\, T(v)$ for all $u, v \in K$ and $\lambda \in \Lambda$. If these inequalities hold with $\geq$, resp. $=$, instead of $\leq$, then the operator $T\colon K \rightarrow F_*$ is called Λ-*concave*, resp. Λ-*affine*. Of course, all these notions are of interest only when $\Lambda \cap (0, 1)$ is non-empty. Finally, we shall use the pointwise order relation for operators from K into the space F_*: for $S, T\colon K \rightarrow F_*$ we write $S \leq T$ on K if $S(x) \leq T(x)$ holds for all $x \in K$.

The following sandwich type theorem extends a basic result from convex analysis to the more general case of Λ-convex operators. Our approach seems to be new even in the classical setting of convex functions on a convex set.

Theorem 1. *For a given subset Λ of $[0, 1]$, let K be a Λ-convex subset of E, and consider a Λ-convex operator $T\colon K \rightarrow F_*$ and a Λ-concave* *operator R: $K \to F_\bullet$. If $R \le T$ on K, then there exists a Λ-affine operator S: $K \to F_\bullet$ such that $R \le S \le T$ on K.*

Proof. If R is constant $= -\infty$, then the assertion follows with the obvious choice $S := -\infty$. Hence we may assume that the set $X := \{u \in K: R(u) > -\infty\}$ is non-empty so that $C := \inf\{T(u) - R(u): u \in X\}$ exists in F and satisfies $C \ge 0$. Now, let $\mathfrak{M}$ consist of all those Λ-convex operators S: $K \to F_\bullet$ which satisfy $R \le S \le T - C$ on K. Note that $\mathfrak{M}$ is non-empty, since $T - C$ belongs to $\mathfrak{M}$. Moreover, $\mathfrak{M}$ is downward inductive in the pointwise order relation, since for every chain $\mathfrak{K}$ in $\mathfrak{M}$ the pointwise infimum over $\mathfrak{K}$ exists by the Dedekind completeness of F and certainly defines a lower bound for $\mathfrak{K}$ in $\mathfrak{M}$. Thus Zorn's lemma yields the existence of minimal operators in $\mathfrak{M}$, and it remains to be seen that the minimal operators in $\mathfrak{M}$ are Λ-affine. To this end, let $S \in \mathfrak{M}$ be minimal, and consider an arbitrary $\lambda \in \Lambda$ with $0 < \lambda < 1$. Then the operator P: $K \to F_\bullet$ given by

$$P(x) := \inf\left\{\frac{1}{\lambda} S(\lambda x + (1 - \lambda)u) - \frac{1 - \lambda}{\lambda} R(u) : u \in X\right\}$$

for all $x \in K$ obviously satisfies $R \le P$ on K, and a simple calculation based on the Λ-convexity of S on K and the Λ-concavity of R on X shows that P is Λ-convex. Finally, from the estimates

$$\begin{aligned}\frac{1}{\lambda} S(\lambda x + (1 - \lambda)u) - \frac{1 - \lambda}{\lambda} R(u) \le S(x) + \frac{1 - \lambda}{\lambda} S(u)&\\ - \frac{1 - \lambda}{\lambda} R(u) \le S(x) + \frac{1 - \lambda}{\lambda} [T(u) - R(u) - C]&\end{aligned}$$

for all $x \in K$ and $u \in X$ and from the definition of C, we infer that $P \le S$ on K. Therefore, the minimality of S in $\mathfrak{M}$ implies that $P = S$ on K and hence that

$$S(\lambda x + (1 - \lambda)u) - \lambda S(x) \ge (1 - \lambda)R(u)$$
$$\text{for all } u, x \in K \text{ with } S(x) > -\infty.$$

These inequalities enable us to show that S is λ-affine. Indeed, given an arbitrary $x \in K$ with $S(x) > -\infty$, let us introduce the operator Q: $K \to F_\bullet$

by

$$Q(y) := \frac{1}{1-\lambda} S(\lambda x + (1-\lambda)y) - \frac{\lambda}{1-\lambda} S(x) \qquad \text{for all } y \in K.$$

Then the preceding estimates guarantee that $R \le Q$ on K. Moreover, it is easily verified that Q is Λ-convex and satisfies $Q \le S$ on K. Again, by the minimality of S in $\mathfrak{M}$, we conclude that $Q = S$ on K and consequently $S(\lambda x + (1-\lambda)y) = \lambda S(x) + (1-\lambda)S(y)$ for all $y \in K$. Also, by the λ-convexity of S, this identity obviously holds for all $x \in X$ with $S(x) = -\infty$. It follows that S is λ-affine for all $\lambda \in \Lambda$, which completes the proof.

COROLLARY 2. *Let $K, L \subseteq E$ be Λ-convex for a given subset Λ of* $[0, 1]$, *assume that $\emptyset \neq L \subseteq K$, and consider a Λ-convex operator $T\colon K \to F_*$. Then, for every Λ-concave operator $R\colon L \to F_*$ with $R \le T$ on L, there exists a Λ- affine operator $S\colon K \to F_*$ such that $R \le S$ on L and $S \le T$ on K. In particular, there is a Λ-affine operator $S\colon K \to F_*$ with $S \le T$ on K such that* $\inf\{S(u)\colon u \in L\} = \inf\{T(u)\colon u \in L\}$.

Proof. We first extend the operator R from L to K by defining $R(x) := -\infty$ for all $x \in K \setminus L$. Then $R\colon K \to F_*$ is Λ-concave and satisfies $R \le T$ on K. Hence the first assertion follows immediately from Theorem 1, and the second assertion follows from the first by taking $R\colon L \to F_*$ to be constant $= \inf\{T(u)\colon u \in L\}$.

The last statement of the preceding result contains the following support theorem as a special case: every Λ-convex operator $T\colon K \to F_*$ on a Λ-convex subset K of E is the pointwise maximum of the Λ-affine operators $S\colon K \to F_*$ which are dominated by T on K. This result will play a key role in our further study of Λ-convex operators.

Given an arbitrary non-empty subset Λ of $[0, 1]$, let $[\Lambda]$ denote the intersection of $[0, 1]$ with the subfield generated by Λ in the real number field $\mathbb{R}$. Thus $[\Lambda]$ contains the rationals $\mathbb{Q} \cap [0, 1]$ and hence is dense in $[0, 1]$. The following theorem shows that, for operators defined on a convex set, the convexity with respect to Λ automatically extends to $[\Lambda]$ as soon as $\Lambda \cap (0, 1)$ is non-empty. In particular, it turns out that non-trivial Λ-convex operators are always rationally convex. This result has been obtained by Kuhn [12] in the scalar-valued case $F = \mathbb{R}$. Let us point out, however, that the approach presented here is not only more general, but also much more direct and self-contained, since we do not have to invoke the sophisticated abstract version of the Hahn–Banach theorem due to Rodé [17].

THEOREM 3. *For an arbitrary subset Λ of* $[0, 1]$ *and every operator T: $K \to F_*$ defined on a $[\Lambda]$-convex subset K of E, we have*:

(a) *If T is Λ-affine and $\Lambda \cap (0, 1)$ is non-empty, then T is $[\Lambda]$-affine.*

(b) *If T is Λ-convex and $\Lambda \cap (0, 1)$ is non-empty, then T is $[\Lambda]$-convex.*

Note that assertion (a) may be considered as a special case of assertion (b), since the Λ-affine and the Λ-convex operators from K into F_* certainly coincide, if F is endowed with the trivial Dedekind complete order given by equality $=$. But, actually, assertion (a) is an elementary fact, which has to be combined with our sandwich theory to establish part (b).

Proof. To prove assertion (a), let us define $A(T)$ to be the set of all $\alpha \in [\Lambda]$ for which the identity $T(\alpha u + (1 - \alpha)v) = \alpha T(u) + (1 - \alpha)T(v)$ holds for all $u, v \in K$. Then clearly $\Lambda \cup \{0, 1\} \subseteq A(T) \subseteq [\Lambda]$, and the following series of arguments will show that actually $A(T) = [\Lambda]$. First, it is easily seen that $\alpha\beta + (1 - \alpha)\gamma \in A(T)$ for all $\alpha, \beta, \gamma \in A(T)$. In particular, this observation shows that

$$\alpha, \beta \in A(T) \text{ implies } 1 - \alpha \in A(T) \text{ and } \alpha\beta \in A(T). \tag{1}$$

Now the crucial step is to establish that

$$\alpha, \beta \in A(T) \qquad \text{with } \alpha \leq \beta \text{ and } \beta \neq 0 \text{ implies } \frac{\alpha}{\beta} \in A(T). \tag{2}$$

To prove this assertion, we may assume $\alpha < \beta$ so that $\gamma := \alpha/\beta \in [0, 1)$. Then, given any $u, v \in K$, let $w := \gamma u + (1 - \gamma)v$. Clearly $w \in K$ by the $[\Lambda]$-convexity of K. Moreover, we have $\beta w + (1 - \beta)v = \alpha u + (1 - \alpha)v$ and hence $\beta T(w) + (1 - \beta)T(v) = \alpha T(u) + (1 - \alpha)T(v)$ because of $\alpha, \beta \in A(T)$. We conclude that

$$T(w) + \left(\frac{1}{\beta} - 1\right) T(v) = \gamma T(u) + \left(\frac{1}{\beta} - \gamma\right) T(v),$$

which implies the desired identity $T(w) = \gamma T(u) + (1 - \gamma)T(v)$ provided that $T(v) \neq -\infty$. In the remaining case $T(v) = -\infty$ we have to show that also $T(w) = -\infty$. To this end, let us fix some $\lambda \in \Lambda \cap (0, 1) \subseteq A(T)$ and choose an integer $k \in \mathbb{N}$ such that $\lambda^k < 1 - \gamma$. From (1) we know that $\mu := 1 - \lambda^k \in A(T)$. Since

$$w = \gamma u + (1 - \gamma)v = \mu z + (1 - \mu)v \qquad \text{with}$$

$$z := \frac{\gamma}{\mu} u + \left(1 - \frac{\gamma}{\mu}\right) v \in K,$$

we arrive at $T(w) = \mu T(z) + (1 - \mu)T(v)$, which shows that $T(v) = -\infty$ implies indeed that $T(w) = -\infty$. Thus $T(\gamma u + (1 - \gamma)v) = \gamma T(u) + (1 - \gamma)T(v)$ holds for arbitrary $u, v \in K$, which completes the proof of (2). We next observe that

$$\alpha, \beta \in A(T) \qquad \text{with } \alpha \leq \beta \text{ implies } \beta - \alpha \in A(T). \tag{3}$$

In fact, since $\beta - \alpha = \beta(1 - \alpha/\beta)$ if $\beta \neq 0$, this assertion is an immediate consequence of (1) and (2). From (3) it follows easily that

$$\alpha, \beta \in A(T) \qquad \text{with } \alpha + \beta \leq 1 \text{ implies } \alpha + \beta \in A(T). \tag{4}$$

Indeed, $\beta \leq 1 - \alpha$ with $1 - \alpha \in A(T)$ implies $1 - \alpha - \beta \in A(T)$ by (3), thus $\alpha + \beta \in A(T)$. We finally claim that the set of quotients $X := \{\pm\alpha/\beta\colon \alpha, \beta \in A(T) \text{ with } \beta > 0\}$ fulfills

$$\frac{\alpha}{\beta} \pm \frac{\gamma}{\delta} \in X \qquad \text{for all } \alpha, \beta, \gamma, \delta \in A(T) \text{ with } \beta, \delta > 0. \tag{5}$$

To prove this assertion, we take again some $\lambda \in \Lambda \cap (0, 1) \subseteq A(T)$ and then choose an integer $k \in \mathbb{N}$ such that $\mu := \lambda^k$ satisfies $|\alpha\delta \pm \beta\gamma|\mu < 1$. Then it is immediately clear from (1), (3), and (4) that $\alpha\delta\mu \pm \beta\gamma\mu \in \pm A(T)$ and hence that

$$\frac{\alpha}{\beta} \pm \frac{\gamma}{\delta} = \frac{\alpha\delta\mu \pm \beta\gamma\mu}{\beta\delta\mu} \in X.$$

Now an obvious combination of (1) and (5) shows that X is a subfield of $\mathbb{R}$, from which we conclude that $[\Lambda] \subseteq [A(T)] \subseteq X \cap [0, 1]$. On the other hand, another application of (2) reveals that $X \cap [0, 1] \subseteq A(T)$. It follows that $[\Lambda] \subseteq A(T)$, which completes the proof of assertion (a). To prove part (b), let $x, y \in K$ and $\lambda \in [\Lambda]$ be arbitrarily given, and consider $u := \lambda x + (1 - \lambda)y \in K$. By Corollary 2, there exists a Λ-affine operator $S\colon K \to F_\bullet$ such that $S \leq T$ on K and $S(u) = T(u)$. From (a) we conclude that $\lambda \in [\Lambda] = A(S)$ and therefore $T(\lambda x + (1 - \lambda)y) = T(u) = S(u) = \lambda S(x) + (1 - \lambda)S(y) \leq \lambda T(x) + (1 - \lambda)T(y)$, which proves that T is indeed $[\Lambda]$-convex. The assertion follows.

It is interesting to note that Theorem 3 is sharp even in the scalar-valued setting $F = \mathbb{R}$. Indeed, it has been shown by Ger [8] that, for every convex set K consisting of more than one point and for every subfield X of $\mathbb{R}$, there exists a function $T\colon K \to X$ with the following two properties: the largest subset Λ of $[0, 1]$ for which T is Λ-affine is precisely $X \cap [0, 1]$, and the largest subset Λ of $[0, 1]$ for which T is Λ-convex is again precisely X

$\cap [0, 1]$. Actually, Ger was only interested in the case of Λ-convexity, but it is clear from the proof of Theorem 1 in [8] that his construction yields a function T: $K \to X$ which is even affine with respect to $X \cap [0, 1]$.

We close this section with the following general version of Jensen's inequality for operators of convex type. Note that the result contains the case of Λ-affine operators as a special case, since we may endow the range space F with the trivial order structure given by equality.

COROLLARY 4. *Let Λ be an arbitrary subset of* $[0, 1]$, *and consider a Λ-convex operator T: $K \to F_*$ on a $[\Lambda]$-convex subset K of E. Then we have*

$$T\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i T(x_i)$$

for all $x_1, \ldots, x_n \in K$ and $\lambda_1, \ldots, \lambda_n \in \Lambda$ with $\sum_{i=1}^{n} \lambda_i = 1$.

The proof of Corollary 4 follows from part (b) of Theorem 3 by an obvious inductive argument. Although the result looks quite elementary, we do not know of any proof of this inequality, which avoids making use of a result like the sandwich or support theorem.

3. THE CASE OF LINEAR OPERATORS

Again, let E be an arbitrary real vector space, and consider an ordered real vector space F. As before, the positive cone F_+ of F is assumed to be proper, but not necessarily generating. Recall that the vector ordering of F is said to be *Archimedean* if $x, y \in F$ with $nx \leq y$ for all $n \in \mathbb{N}$ implies that $x \leq 0$. In other words, this condition means precisely that F_+ is lineally closed. It is well known and easily seen that every Dedekind complete ordered vector space is Archimedean. Indeed, in the indicated situation, $s := \sup\{nx : n \in \mathbb{N}\}$ exists in F and satisfies $2s \leq s$, which implies that $x \leq s \leq 0$. Finally recall that an operator ϑ: $E \to F$ is said to be *sublinear* if $\vartheta(u + v) \leq \vartheta(u) + \vartheta(v)$ and $\vartheta(tu) = t\vartheta(u)$ holds for all $u, v \in E$ and all real $t \geq 0$. The following elementary lemma will be the main vehicle for applications of Theorem 1 to linear functional analysis.

LEMMA 5. *Let ϑ: $E \to F$ be a sublinear operator from E into an Archimedean ordered vector space F, and consider an operator ψ: $E \to F$ such that $\psi \leq \vartheta$ on E. If ψ is λ-affine for some $0 < \lambda < 1$, then ψ is actually affine on E. Moreover, the operator $\varphi := \psi - \psi(0)$ is linear and satisfies $\psi \leq \varphi \leq \vartheta$ on E.*

Proof. Since φ is λ-affine and satisfies $\varphi(0) = 0$, we obtain for all $u, v \in E$ the identities $\lambda(1-\lambda)\varphi(u+v) = \varphi(\lambda(1-\lambda)u + (1-\lambda)\lambda v) = \lambda\varphi((1-\lambda)u) + (1-\lambda)\varphi(\lambda v) = \lambda(1-\lambda)(\varphi(u) + \varphi(v))$ and therefore $\varphi(u+v) = \varphi(u) + \varphi(v)$. Thus $\varphi\colon E \to F$ is additive and therefore $\mathbb{Q}$-linear. Moreover, since $\psi(0) \leq \vartheta(0) = 0$, we have $\psi \leq \varphi$ on E. Next, to prove that $\varphi \leq \vartheta$ on E, we use the additivity of φ to arrive at $n(\varphi(x) - \vartheta(x)) = \varphi(nx) - \vartheta(nx) = \psi(nx) - \psi(0) - \vartheta(nx) \leq -\psi(0)$ for all $n \in \mathbb{N}$ and $x \in E$. Since the ordering of F is Archimedean, this implies that $\varphi(x) \leq \vartheta(x)$ for all $x \in E$. Finally, to show that φ is homogeneous over $\mathbb{R}$, we consider an arbitrary real number $t > 0$ and choose rationals numbers $t_n < t$ such that $s_n := t - t_n \to 0$ as $n \to \infty$. Then, for each $x \in E$, we have $\varphi(tx) - t\varphi(x) = \varphi(s_n x) - s_n\varphi(x) \leq \vartheta(s_n x) - s_n\varphi(x) = s_n(\vartheta(x) - \varphi(x))$ for all $n \in \mathbb{N}$. Since $\varphi(x) \leq \vartheta(x)$ and F is Archimedean, it follows that $\varphi(tx) \leq t\varphi(x)$. In view of $\varphi(-x) = -\varphi(x)$, we conclude that $\varphi(tx) = t\varphi(x)$ for all $x \in E$ and $t \in \mathbb{R}$. This shows that φ is linear and hence that $\psi = \varphi + \psi(0)$ is affine.

THEOREM 6. *Let $\vartheta\colon E \to F$ be a sublinear operator from E into a Dedekind complete ordered vector space F, and consider an operator $\tau\colon E \to F$ such that $\tau \leq \vartheta$ on E. Moreover, assume that there exists a pair of real numbers $\alpha, \beta > 0$ such that τ is (α, β)-concave in the sense that $\tau(\alpha u + \beta v) \geq \alpha\tau(u) + \beta\tau(v)$ for all $u, v \in E$. Then there exists a linear operator $\varphi\colon E \to F$ such that $\tau \leq \varphi \leq \vartheta$ on E.*

Proof. In light of Theorem 1 and Lemma 5, it remains to find an operator $\rho\colon E \to F$ which is λ-concave for some $0 < \lambda < 1$ and satisfies $\tau \leq \rho \leq \vartheta$ on E. Our canonical choice is $\lambda := \alpha/(\alpha + \beta)$ and $\rho(x) := \sup\{(\alpha + \beta)^{-n}\tau((\alpha + \beta)^n x)\colon n \in \mathbb{N}\}$ for all $x \in E$. Obviously, $\rho\colon E \to F$ is well-defined and fulfills $\tau \leq \rho \leq \vartheta$ on E. Moreover, since τ is (a, β)-concave and therefore satisfies $(\alpha + \beta)^{-n}\tau((\alpha + \beta)^n x) \leq (\alpha + \beta)^{-m}\tau((\alpha + \beta)^m x)$ for all $x \in E$ and all $n, m \in \mathbb{N}$ with $n \leq m$, it is easily verified that ρ is λ-concave. The assertion follows.

THEOREM 7. *Let $\vartheta\colon E \to F$ be a sublinear operator from E into a Dedekind complete ordered vector space F, and consider an arbitrary operator $\sigma\colon S \to F$ on a non-empty subset S of E such that $\sigma \leq \vartheta$ on S. Moreover, assume that there exists a pair of real numbers $\alpha, \beta > 0$ and some $z \in F_+$ such that the following condition is fulfilled:*

$$\begin{gathered}\textit{For all } u, v \in S \textit{ and } \varepsilon > 0 \textit{ there is some } w \in S \textit{ such that}\\ \vartheta(w - \alpha u - \beta v) \leq \sigma(w) - \alpha\sigma(u) - \beta\sigma(v) + \varepsilon z.\end{gathered} \tag{6}$$

Then there exists a linear operator $\varphi\colon E \to F$ such that $\sigma \leq \varphi$ on S and $\varphi \leq \vartheta$ on E.

Proof. Let $\tau(x) := \sup\{\sigma(u) - \vartheta(u - x)\colon u \in S\}$ for all $x \in E$. Then it is immediately clear that $\tau\colon E \to F$ is well-defined and fulfills $\sigma \leq \tau$ on S as well as $\tau \leq \vartheta$ on E. Moreover, since F is Archimedean, an elementary computation shows that condition (6) forces τ to be (α, β)-concave on E. Hence the assertion follows from Theorem 6.

Notice that condition (6) is fulfilled if, for some pair of real numbers α, $\beta > 0$, the set S satisfies $\alpha S + \beta S \subseteq S$ and the operator σ is (α, β)-concave on S. This shows that the preceding two theorems are basically equivalent. In the scalar-valued setting $F = \mathbb{R}$, Theorem 7 reduces to the main result of Fuchssteiner and König [5]. Note, however, that their proof is completely different from ours and does not extend to the vector-valued case. Another very different approach to Theorem 7 has recently been given in our paper [15], which contains also a detailed discussion of the weak convexity condition (6). Conditions of this kind date back to classical work by Ky Fan and Heinz König on minimax and Hahn–Banach type theorems and have proven to be extremely useful in different branches of analysis. We refer to [5, 10, 11,13, 15] for a number of applications and further references.

As a straightforward consequence of Theorem 7, we obtain the following vector-valued version of the classical Mazur–Orlicz theorem [14]: if $\vartheta\colon E \to F$ is sublinear from E into a Dedekind complete ordered vector space F and if $\sigma\colon S \to F$ denotes an arbitrary operator on a non-empty subset S of E, then there exists a linear operator $\varphi\colon E \to F$ with $\sigma \leq \varphi$ on S and $\varphi \leq \vartheta$ on E if and only if the Mazur–Orlicz condition

$$\sum_{i=1}^{n} t_i\sigma(u_i) \leq \vartheta\left(\sum_{i=1}^{n} t_i u_i\right)$$

$$\text{for all } u_1, \ldots, u_n \in S \text{ and } t_1, \ldots, t_n \geq 0 \text{ with } \sum_{i=1}^{n} t_i = 1$$

is fulfilled. Indeed, necessity is obvious, and sufficiency follows easily from Theorem 7, applied to the concave operator $\tilde{\sigma}$ on the convex hull $\mathrm{co}(S)$ of S, given by

$$\tilde{\sigma}(u) := \sup \left\{ \sum_{i=1}^{n} t_i\sigma(u_i) \colon u = \sum_{i=1}^{n} t_i u_i \right.$$

$$\left. \text{for } u_1, \ldots, u_n \in S,\ t_1, \ldots, t_n \geq 0 \text{ with } \sum_{i=1}^{n} t_i = 1 \right\}$$

for all $u \in \mathrm{co}(S)$. Thus the role of the convexity assumption (6) is to ensure that the Mazur–Orlicz consistency condition can be replaced by the much simpler condition $\sigma \leq \vartheta$ on S.

Theorem 7 also yields the following two generalizations of a recent result due to Martellotti and Salvadori [13]. Again, let $\vartheta: E \rightarrow F$ be a sublinear operator into a Dedekind complete ordered vector space F, and consider a non-empty set $S \subseteq E$. If $\vartheta \geq 0$ on S and if there exist $\alpha, \beta > 0$ and $z \in F_+$ such that, for all $u, v \in S$ and $\varepsilon > 0$, there is some $w \in S$ with $\vartheta(w - \alpha u - \beta v) \leq \varepsilon z$, then there exists a linear operator $\varphi: E \rightarrow F$ such that $\varphi \leq \vartheta$ on E and $\varphi \geq 0$ on S. Moreover, if there exist $0 < \lambda < 1$ and $z \in F_+$ such that, for all $u, v \in S$ and $\varepsilon > 0$, there is some $w \in S$ with $\vartheta(w - \lambda u - (1 - \lambda)v) \leq \varepsilon z$, then there exists a linear operator $\varphi: E \rightarrow F$ such that $\varphi \leq \vartheta$ on E and $\inf\{\varphi(u): u \in S\} = \inf\{\vartheta(u): u \in S\}$. Note that the main theorem in [13] deals only with the much simpler case $\lambda = \frac{1}{2}$, $z = 0$ and that the technicalities are completely different. An application of this special case to the theory of lattice-valued capacities may be found in [1].

4. Applications to Boundaries and Representing Measures

In this final section, the preceding sandwich theory will be used in the scalar-valued setting $F = \mathbb{R}$ to derive some rather general results on representing measures and boundaries for families of upper semicontinuous functions on a compact Hausdorff space X. Let $C(X)$ denote the real vector space of all real-valued continuous functions on X, and consider the sublinear functional $\vartheta: C(X) \rightarrow \mathbb{R}$ given by $\vartheta(f) := \max(f)$ for all $f \in C(X)$. As usual, the set $\operatorname{Prob}(X)$ of Radon probability measures on X is defined to consist of all linear functionals μ on $C(X)$ which are dominated by ϑ in the sense that $\mu(f) \leq \max(f)$ for all $f \in C(X)$. It is easily seen that these are precisely the linear functionals $\mu: C(X) \rightarrow \mathbb{R}$ which satisfy $\mu(1) = 1$ and $\mu(f) \geq 0$ whenever $f \geq 0$. Hence, by the Riesz representation theorem, the $\mu \in \operatorname{Prob}(X)$ are, via integration, in one-to-one correspondence with the regular Borel probability measures on X. It will be convenient to identify the corresponding Radon and regular Borel measures on X and to write $\mu(f) = \int f\, d\mu$ for $\mu \in \operatorname{Prob}(X)$ and $f \in C(X)$.

We shall also need the following standard extension procedure. Let $\operatorname{USC}(X)$ denote the convex cone of all upper semicontinuous functions $f: X \rightarrow [-\infty, \infty)$. Then, for any $\mu \in \operatorname{Prob}(X)$, it is natural to define $\mu(f) := \int f\, d\mu = \inf\{\int g\, d\mu: g \in C(X) \text{ with } g \geq f \text{ on } X\} \geq -\infty$ for all $f \in \operatorname{USC}(X)$. Finally, given a closed subset K of X, let us introduce the sublinear functional $\vartheta_K: C(X) \rightarrow \mathbb{R}$ by $\vartheta_K(f) := \max(f|K)$ for all $f \in C(X)$. Obviously $\vartheta_K \leq \vartheta$ on $C(X)$, and it is not hard to characterize the linear functionals μ on $C(X)$ dominated by ϑ_K. Namely, we have $\mu \leq \vartheta_K$ on $C(X)$ if and only if the measure μ is supported by K, which in turn is equivalent to $\mu(\chi_K) = 1$, where χ_K denotes the characteristic function of K. For more details on these elementary facts and a weaker version of the following result we refer to [10].

THEOREM 8. *Let K be a non-empty closed subset of a compact Hausdorff space X, and consider a family $T \subseteq \mathrm{USC}(X)$ such that $\max(f|K) \geq 0$ for all $f \in T$. Moreover, assume that there exists a pair of real numbers α, $\beta > 0$ such that*

$$\begin{gathered} \text{for all } f, g \in T \text{ and } \varepsilon > 0 \text{ there is an } h \in T \\ \text{such that } h \leq \alpha f + \beta g + \varepsilon \text{ on } K. \end{gathered} \tag{7}$$

Then there exists some $\mu \in \mathrm{Prob}(X)$ such that $\mathrm{supp}(\mu) \subseteq K$ and $\int f\, d\mu \geq 0$ for all $f \in T$.

Proof. Consider the sublinear functional ϑ_K on the vector space $E = C(X)$ and the functional $\sigma \equiv 0$ on the set $S := \{g \in C(X)\colon g \geq f \text{ on } K \text{ for some } f \in T\} \subseteq E$. Then the result follows easily from Theorem 7 and the remarks preceding Theorem 8.

Note that the weak convexity condition (7) is fulfilled if $\alpha T + \beta T \subseteq T$ holds for some pair of real numbers α, $\beta > 0$. In particular, Theorem 8 applies whenever T is convex or satisfies the cone condition $T + T \subseteq T$. As an immediate special case, we obtain a classical result on the existence of Jensen measures, cf. Theorem 2.2.4 of [7]. Indeed, if ϕ is a multiplicative linear functional on a uniform algebra A on X, then the family $T := \{\log |f| - \log |\phi(f)|\colon f \in A \text{ with } \phi(f) \neq 0\} \subseteq \mathrm{USC}(X)$ satisfies both $T + T \subseteq T$ and $\max(f) \geq 0$ for all $f \in T$. Hence Theorem 8 yields the existence of some $\mu \in \mathrm{Prob}(X)$ with $\log |\phi(f)| \leq \int \log |f|\, d\mu$ for all $f \in A$, which means precisely that μ is a Jensen measure for ϕ.

Next, recall that a subset K of X is said to be a *boundary* for a non-empty family $T \subseteq \mathrm{USC}(X)$, if every $f \in T$ assumes its supremum on X at some point of K. If T separates the points of X, then, by Bauer's maximum principle [2], an important example is furnished by the *Choquet boundary* $\mathrm{Ch}(X, T)$, which is defined to consist of all points $u \in X$ such that the Dirac measure at u is the only measure $\mu \in \mathrm{Prob}(X)$ which satisfies $f(u) \leq \int f\, d\mu$ for all $f \in T$. In general, $\mathrm{Ch}(X, T)$ need not be the smallest boundary, actually neither a smallest boundary nor a smallest closed boundary will exist in general; see for instance Chapter II.3 of [6] for further information. If there does exist a smallest closed boundary for T, then this is called the *Shilov boundary* for T and denoted by $\mathrm{Sh}(X, T)$. From Theorem 8 we obtain immediately the following general theorem on the existence and description of the Shilov boundary; see also [2, 3, 6, 16] for some related results.

THEOREM 9. *Let $T \subseteq \mathrm{USC}(X)$ separate the points of X, and assume that for each $u \in X$ there exists a pair of real numbers α, $\beta > 0$ with the following convexity property:*

For all $f, g \in T$ *with* $f(u), g(u) > -\infty$ *and all* $\varepsilon > 0$, *there is some* $h \in T$ *such that* $h(u) > -\infty$ *and* $h - h(u) \leq \alpha[f - f(u)] + \beta[g - g(u)] + \varepsilon$ *on* X. (8)

Then $\mathrm{Sh}(X, T)$ *exists and is equal to the closure of the Choquet boundary* $\mathrm{Ch}(X, T)$.

Proof. In view of Bauer's maximum principle [2], it suffices to show that every closed boundary K for T necessarily contains $\mathrm{Ch}(X, T)$. Given an arbitrary $u \in \mathrm{Ch}(X, T)$, let us introduce the family $T_u := \{f - f(u): f \in T \text{ with } f(u) > -\infty\} \neq \emptyset$. Because of condition (8) we may apply Theorem 8 to T_u. Hence there is a $\mu \in \mathrm{Prob}(X)$ supported by K such that $f(u) \leq \int f\, d\mu$ holds for all $f \in T$. Since $u \in \mathrm{Ch}(X, T)$, this implies that μ is the Dirac measure at u. From $\mathrm{supp}(\mu) \subseteq K$ we finally conclude that u belongs to K. Thus $\mathrm{Ch}(X, T) \subseteq K$.

Again, it should be noted that the fairly mild convexity assumption (8) on T is fulfilled, if $\alpha T + \beta T \subseteq T$ holds for some pair of real numbers $\alpha, \beta > 0$ and, in particular, if T is convex or closed under addition. Hence the preceding result contains Bauer's celebrated theorem on the existence of the Shilov boundary as a special case; see Satz 2 of [2] and also [6]. Let us state another typical application. As usual, $C(X, \mathbb{C})$ stands for the complex vector space of all complex-valued continuous functions on X.

COROLLARY 10. *Let* $A \subseteq C(X, \mathbb{C})$ *be a complex-linear subspace containing the complex constants* $\mathbb{C}$, *and assume that* A *separates the points of* X. *Then, for each of the following three families*

$$|A| := \{|f|: f \in A\}, \qquad \operatorname{Re} A := \{\operatorname{Re} f: f \in A\},$$
$$A^x := \{|f_1^2| + \cdots + |f_n^2|: f_1, \ldots, f_n \in A\},$$

the Shilov boundary exists and is equal to the closure of the corresponding Choquet boundary. Moreover, we have $\mathrm{Sh}(X, |A|) = \mathrm{Sh}(X, \operatorname{Re} A) = \mathrm{Sh}(X, A^x)$.

Proof. (i) First, given any $u, v \in X$ with $u \neq v$, we have $f(u) \neq f(v)$ for some $f \in A$. Since $g := |f - f(u)|$ belongs to $|A|$ and satisfies $g(u) \neq g(v)$, it follows that $|A|$ separates the points of X. Further, given $u \in X$ and $f, g \in A$, we choose some $\gamma \in \mathbb{C}$ with $|\gamma| = 1$ such that $|f(u) + \gamma g(u)| = |f(u)| + |g(u)|$ and let $h := f + \gamma g \in A$. Then clearly $|h| - |h(u)| \leq |f| - |f(u)| + |g| - |g(u)|$ on X. With the choice $\alpha = \beta = 1$, it follows that $|A|$ satisfies condition (8) of Theorem 9. Consequently, $\mathrm{Sh}(X, |A|)$ does exist and coincides with the closure of $\mathrm{Ch}(X, |A|)$. A similar application of Theorem 9 yields the corresponding results for $\operatorname{Re} A$ and A^x, since both families are closed under addition. (ii) To prove the final statement, we note that

$\mathrm{Sh}(X, |A|)$ is equal to the closure of $\mathrm{Ch}(X, \mathrm{Re}\, A)$ by Proposition 6.4 of [16]. Together with part (i) we conclude that $\mathrm{Sh}(X, |A|) = \mathrm{Sh}(X, \mathrm{Re}\, A)$. To establish the remaining identity $\mathrm{Sh}(X, |A|) = \mathrm{Sh}(X, A^x)$, we shall show that $|A|$ and A^x have the same boundaries. Clearly, every boundary for A^x is a boundary for $|A|$. Conversely, assume that $K \subseteq X$ is a boundary for $|A|$ and consider an arbitrary function $f \in A^x$, i.e.,

$$f = |f_1^2| + \cdots + |f_n^2| \qquad \text{with } f_1, \ldots, f_n \in A.$$

Let $M := \max(f)^{1/2}$, fix an $x_o \in X$ such that $f(x_o) = M^2$, choose a unitary $n \times n$ matrix $U = (u_{ij})$ such that $(M, 0, \ldots, 0) = (f_1(x_o), \ldots, f_n(x_o))U$, and put

$$g := |g_1^2| + \cdots + |g_n^2| \in A^x, \qquad \text{where}$$

$$g_j := \sum_{i=1}^{n} f_i u_{ij} \in A \text{ for } j = 1, \ldots, n.$$

Then $|g_1(x)|^2 \leq g(x) = f(x) \leq f(x_o) = M^2 = |g_1(x_o)|^2$ for all $x \in X$, since unitary matrices preserve the Euclidean norm. Hence, since K is a boundary for $|A|$, there is some $z_o \in K$ such that $|g_1(x_o)| = |g_1(z_o)|$. We conclude that

$$f(z_o) = g(z_o) \geq |g_1(z_o)|^2 = M^2,$$

which shows that K is indeed a boundary for A^x. The assertion follows.

If $A \subseteq C(X, \mathbb{C})$ happens to be a complex subalgebra which separates the points of X and contains all complex constants, then the first part of the preceding corollary reduces to a well-known result due to Shilov; see for instance [2]. Actually, in this particular case the family $T := \{\log |f| : f \in A\}$ satisfies $T + T \subseteq T$ so that the assertion on $\mathrm{Sh}(X, |A|)$ can be easily deduced from Bauer's theorem on the existence of the Shilov boundary; for details we refer to Satz 18 of [2]. However, if A is not closed under multiplication, then this method obviously fails. Hence it is interesting that our sandwich theory is appropriate to cover the general case of subspaces instead of subalgebras.

Corollary 10 applies, for instance, to the complex-linear space A of all continuous complex-valued affine functionals on a compact convex subset X of a complex locally convex space. It follows that, in this situation, the Shilov boundaries for $|A|$, $\mathrm{Re}\, A$, and A^x exist and coincide with the closure of the extreme points of X. This generalizes recent results on Shilov boundaries due to Kajetanowicz [9] and Tonev [18]. As explained in [18], the class A^x arises naturally in the context of minimal multipletuple boundaries. We conclude with another typical application of Theorem 8 in Choquet theory.

THEOREM 11. *Given an arbitrary family* $T \subseteq \mathrm{USC}(X)$, *assume that for each* $u \in X$ *the convexity condition* (8) *is fulfilled, and let* K *be a closed boundary for* T. *Then, for every* $\lambda \in \mathrm{Prob}(X)$, *there exists a* $\mu \in \mathrm{Prob}(X)$ *such that* $\mathrm{supp}(\mu) \subseteq K$ *and* $\int f\, d\lambda \leq \int f\, d\mu$ *for all* $f \in \mathrm{MC(T)}$, *where* $\mathrm{MC}(T)$ *denotes the maximum-stable convex cone generated by* T *and the real constants* $\mathbb{R}$.

Notice that the last inequalities mean precisely that λ is dominated by μ in the Choquet order and that the result applies, in particular, to the Shilov boundary $K = \mathrm{Sh}(X, T)$ whenever T separates the points of X. Thus Theorem 11 may be considered as a general version of Choquet's representation theorem in the case of closed boundaries; see [16] and also [6] for a more recent account of Choquet theory.

Proof. We first show that K is even a boundary for the enlarged family $\mathrm{MC}(T)$. This fact could be obtained by elementary, but technically rather involved approximation arguments. Instead, we prefer to give an illuminating short proof based on Theorem 8. Let us fix an arbitrary $u \in X$ and define $T_u := \{f - f(u): f \in T \text{ with } f(u) > -\infty\}$. If T_u is non-empty, then Theorem 8 supplies us with some $\nu \in \mathrm{Prob}(X)$ supported by K such that $f(u) \leq \int f\, d\nu$ for all $f \in T$. If T_u happens to be empty, we arrive at the same conclusion simply by taking ν to be the Dirac measure at some point of K. Next observe that each $g \in \mathrm{MC}(T)$ can be written as the pointwise maximum $g = \max(g_1, \ldots, g_k)$ of finitely many functions $g_1, \ldots, g_k$, where each g_j is a finite sum of the form $\alpha_1 f_1 + \cdots + \alpha_n f_n$ with real numbers $\alpha_1, \ldots, \alpha_n \geq 0$ and functions $f_1, \ldots, f_n \in T \cup \mathbb{R}$. Hence it follows that the inequalities $g(u) \leq \int g\, d\nu$ hold for all $g \in \mathrm{MC}(T)$. Because of $\mathrm{supp}(\nu) \subseteq K$, we conclude that $g(u) \leq \max(g|K)$ for all $g \in \mathrm{MC}(T)$. Since $u \in X$ was arbitrarily given, this proves that K is indeed a boundary for $\mathrm{MC}(T)$. Now the proof can be easily completed by another application of Theorem 8. Clearly, the family $T_\lambda := \{g - \int g\, d\lambda: g \in \mathrm{MC}(T) \text{ with } \int g\, d\lambda > -\infty\}$ is non-empty and satisfies $T_\lambda + T_\lambda \subseteq T_\lambda$. Moreover, since K is a boundary for $\mathrm{MC}(T)$, it is clear that $\max(h|K) \geq 0$ for all $h \in T_\lambda$. Hence the assertion follows immediately from Theorem 8.

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